Journal of Applied Mathematics and Mechanics 74 (2010) 108–121

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

The decomposition method for a control problem for an underactuated Lagrangian system☆

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ARTICLE INFO

Article history:
Received 24 April 2009

ABSTRACT

The control problem for an underactuated Lagrangian system is considered. A system of smooth non-linear functions of the generalized coordinates is introduced into the treatment and the number of functions is equal to the number of generalized control forces. The aim of the control is to bring the system in a finite time to a terminal set specified by the level lines of the selected functions, and it is required that the motion at the terminal instant occurs along the level lines. As a result, a development and extension of Chernous'ko's decomposition method is given. This method was proposed for designing feedback control for Lagrangian systems when the number of controls in a system is equal to the number of its degrees of freedom.

1. Formulation of the problem

A system is considered, the dynamics of which are described by differential equations in Lagrangian form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = U_i + Q_i, \quad i = 1,\dots,n \tag{1.1}$$

Here, q_i are the generalized coordinates of the system, U_i are the generalized control forces, Q_i are all the remaining generalized forces including uncontrolled perturbations, n is the number of degrees of freedom of the system, derivatives with respect to time t are denoted by a dot and $T(q, \dot{q})$ is the kinetic energy of the system, specified in the form of a positive-definite quadratic form of the generalized velocities $\dot{q}_i$:

$$T(q,\dot{q}) = \frac{1}{2}\sum_{j,k=1}^{n} a_{jk}(q)\dot{q}_j\dot{q}_k \tag{1.2}$$

We shall assume that all the motions of system (1.1) considered occur in a certain domain D_q in the n-dimensional space R^n such that $q \in D_q$ always. In particular, the domain D_q can be identical to R^n.

It is assumed that geometric constraints of the form

$$|U_i| \le U_i^0, \quad i = 1,\dots,n \tag{1.3}$$

where U_i^0 are given constants, are imposed on the control actions at each instant.

Decomposition method. We will now briefly describe the decomposition method[1] which was proposed in order to bring system (1.1) into a specified state

$$q(\tau) = q^*, \quad \dot{q}(\tau) = 0$$

in a finite time τ, when all the constants U_i^0 are positive. We substitute the expression for the kinetic energy (1.2) into Eqs (1.1) and obtain

$$A(q)\ddot{q} = U + S(q,\dot{q},t) \tag{1.4}$$

☆ *Prikl. Mat. Mekh.* Vol. 74, No. 1, pp. 151-169, 2010.

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0021-8928/$ – see front matter
doi:10.1016/j.jappmathmech.2010.03.011

Here $q=(q_1,\ldots,q_n)$ is the vector of the generalized coordinates of the system, $A(q)$ is the symmetric positive-definite matrix of the kinetic energy of the systems with elements a_{jk} which depend on q, $U=(U_1,\ldots,U_n)$ is the n-dimensional vector of the generalized control forces, $S=(S_1,\ldots,S_n)$ is the vector function

$$S(q,\dot{q},t) = Q(q,\dot{q},t) - \sum_{j,k=1}^{n} \Gamma_{jk}\dot{q}_j\dot{q}_k \tag{1.5}$$

where $\Gamma_{jk}=(\Gamma_{1jk},\ldots,\Gamma_{njk})$ are n-dimensional vectors with the components

$$\Gamma_{ijk} = \frac{\partial a_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial a_{jk}}{\partial q_i} \tag{1.6}$$

and $Q=(Q_1,\ldots,Q_n)$ is an n-dimensional vector which includes the non-controlling generalized forces. It is assumed[1] that the forces Q are dissipative and sufficiently small at low velocities.

Initially, using a control which is directed against the velocity vector, the system is brought into the domain of low velocities where the components of the vector S become small (in modulus) compared with the components of the vector U.

A control is designed using a game approach in the domain of low velocities.

The initial equations are represented in the form

$$\ddot{q} = u + \upsilon; \quad u = A^{-1}(q)U, \quad \upsilon = A^{-1}(q)S \tag{1.7}$$

Here, u plays the role of a new control vector and υ is considered as a perturbation vector. In order to satisfy initial constraints (1.3) imposed on the vector U, the need arises to impose the constraints on the new control u:

$$|u_i| \le u_0, \quad u_0 = (a^*)^{-1} n^{-1/2} \min_i U_i^0, \quad i = 1,\ldots,n \tag{1.8}$$

Here, a^* is a constant which sets an upper bound on the eigenvalues of the matrix $A(q)$ for all q. In other words, if the above constant a^* is known and the constraints (1.8) are satisfied, then satisfaction of condition (1.3) is guaranteed.

The initial problem reduces to the problem of bringing each of the n linear systems (1.7) into a specified terminal state with a zero velocity in a finite time and retaining them there. In the i-th equation of (1.7), the quantity u_i is considered as the control parameter of one player and the quantity υ_i is considered as the counteraction of the second player (the opponent). Conditions have been derived[1] for which the resources of the controlling player prove to be greater than the resources of the opponent:

$$|\upsilon_i| \le \upsilon_0, \quad \upsilon_0 < u_0, \quad i = 1,\ldots,n \tag{1.9}$$

and this game problem has a solution. If the approach of the theory of differential games[2] is applied to the linear systems (1.7) with constraints (1.8) and (1.9), we obtain the expressions $u_i(q_i, \dot{q}_i)$ for designing the guaranteed feedback control which solve the problem when $\upsilon_0 < u_0$.

This decomposition method has been used by other authors.[3,4] It has been extended[5] to the case of non-zero terminal velocities. The decomposition method has also been extended to the problem of tracking a specified trajectory of motion of a Lagrangian system.[6] Subsequent investigations have shown[7] that the decomposition method can also be used to control a double pendulum, a Lagrangian system in which the number of controls is less than the number of degrees of freedom.

A procedure, similar to the decomposition method, was used[8–10] for a special case, that is, for solving the problem of the reorientation of a rigid body when there is disturbance. Another method of controlling dynamical systems of the form (1.1) with constraints (1.3), based on the idea of decomposition, has been proposed[11,12] and other conditions for the realizability of the decomposition method given, the kinetic energy matrix has been assumed to be unknown and the time at which systems are brought into a specified state occurs in a finite time. The control algorithm is proved using the second Lyapunov method. The papers[13–16] touch on the papers.[11–12]

We will now obtain an extension of the decomposition method[1] to the case of underactuated system (1.4), that is, when, in constraints (1.3), some of the constants U_i^0 together with the corresponding functions U_i are equal to zero. We will consider the situation when only the first m ($m<n$) components of the vector U can be non-zero, and we will therefore represent the control vector in the form

$$U = (U_1,\ldots,U_m,0,\ldots,0); \quad |U_i| \le U_i^0, \quad i = 1,\ldots,m \tag{1.10}$$

where U_i^0 are given constants.

The constraint

$$|Q_i| \le Q_i^0, \quad i = 1,\ldots,n \tag{1.11}$$

where $Q_i^0 > 0$ are given constants, is also imposed on the vector of the non-controlling forces.

We introduce the notation

$$x = L(q), \quad \dot{x} = B(q)\dot{q} \tag{1.12}$$

where L is a smooth m-dimensional vector function which depends on q, and B is an $m \times n$ matrix with elements

$$B_{ij} = \partial L_i / \partial q_j$$

The initial state of system (1.4) is given:

$$q(0) = q^0, \quad \dot{q}(0) = \dot{q}^0 \tag{1.13}$$

The initial values of the vector x and its velocity $\dot{x}$, which are given by relations (1.12)

$$x^0 = L(q^0), \quad \dot{x}^0 = B(q^0)\dot{q}^0 \tag{1.14}$$

correspond to it.

We shall subsequently bring system (1.4) into a set determined by the motion of the system along the level lines of the functions $L_i(q)$ ($i = 1,\ldots,m$):

$$x = x^* = \text{const}, \quad \dot{x} = 0 \tag{1.15}$$

From equalities (1.12) and (1.15), we obtain the relations for the terminal q, $\dot{q}$ in the form

$$L(q) = x^*, \quad B(q)\dot{q} = 0 \tag{1.16}$$

We now consider the following control problem.

Problem 1. It is required to construct a feedback control $U(q, \dot{q})$ which satisfies constraint (1.10) and which brings system (1.4) from the state (1.13) into the terminal set (1.16), where x^* is a given constant n-dimensional vector. The time of the control process τ is finite and is not fixed. Without loss in generality, the initial instant of time is taken as being equal to zero.

Problem 1 will be solved with additional simplifying assumptions which are formulated in the following section.

2. Simplifying assumptions

We will first make several simplifying assumptions regarding the kinetic energy matrix. We will assume that

$$\left|\frac{\partial a_{jk}}{\partial q_i}\right| \le c, \quad c = \text{const} > 0, \quad i, j, k = 1,\ldots,n \tag{2.1}$$

and that the eigenvalues of the matrix $A(q)$ for all q lie in the closed interval ($0 < a_* < a^*$). The conditions

$$a_*|z| \le |Az| \le a^*|z|, \quad (a^*)^{-1}|z| \le |A^{-1}z| \le a_*^{-1}|z| \tag{2.2}$$

are therefore satisfied for any n-dimensional vector.

We represent the matrices A and A^{-1} in partitioned form

$$A = \begin{bmatrix} A_1 & \tilde{A}^T \\ \tilde{A} & A_2 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} \alpha_1 & \tilde{\alpha}^T \\ \tilde{\alpha} & \alpha_2 \end{bmatrix} \tag{2.3}$$

Here, $A_1, A_2, \alpha_1, \alpha_2$ are symmetric positive-definite matrices, where A_1 and α_1 are $m \times m$ matrices, A_2 and α_2 are $(n-m) \times (n-m)$ matrices and $\tilde{A}$ and $\tilde{\alpha}$ are $(n-m) \times m$ matrices. It follows from equalities (2.2) that, for all q, the eigenvalues of the matrices $A_1(q)$ and $A_2(q)$ also lie in the interval $[a_*, a^*]$ and that the eigenvalues of the matrices $\alpha_1(q)$ and $\alpha_2(q)$ lie in the interval $[(a^*)^{-1}, a_*^{-1}]$, and, moreover, the following estimate of the norms of the matrices $\tilde{A}(q)$ and $\tilde{\alpha}(q)$ holds

$$\|\tilde{A}(q)\| \le a^*, \quad \|\tilde{\alpha}(q)\| \le (a_*)^{-1}$$

We shall henceforth denote the Euclidean norm of a matrix, that is, the norm of the corresponding linear operator in Euclidean space by $\|Z\|$:

$$\|Z\| = \max_{|z|=1} |Zz|$$

where z is a vector of corresponding dimension. If the matrix Z is symmetric, its norm is equal to the greatest absolute eigenvalue of the matrix. If the matrix Z is asymmetric, its norm is equal to the square root of the greatest eigenvalue of the negative-definite matrix Z^TZ.

We will now make assumptions regarding the matrix B which contains the first partial derivatives of the functions L_i. We will assume that the matrix B can be represented in the form

$$B = \begin{bmatrix} B_1 & \tilde{B} \end{bmatrix} \tag{2.4}$$

where B_1 is a non-singular $m \times m$ matrix for all q. We will assume that the norms of the matrices B_1 and B_l^{-1} are bounded:

$$\|B_1\| \le b^*, \quad \|B_1^{-1}\| \le b_*^{-1} \tag{2.5}$$

where b_* and b^* are positive constants. Hence, the inequalities

$$b_*|z| \le |B_1 z| \le b^*|z|, \quad (b^*)^{-1}|z| \le |B_1^{-1}z| \le b_*^{-1}|z| \tag{2.6}$$

hold for any n-dimensional vector z.

We will also assume that an upper estimate of the norm of the matrix $\tilde{B}$

$$\|\tilde{B}\| \le \tilde{b} \tag{2.7}$$

is known.

We introduce the notation

$$C = BA^{-1}, \quad D = B_1 - \tilde{B}A_2^{-1}\tilde{A} \tag{2.8}$$

It follows from inequalities (2.2), (2.5) and (2.7) that the norm of the matrix C is bounded for all q

$$\|C\| \le c^* \tag{2.9}$$

We represent the matrix C in the form

$$C = \left[C_1 \, \tilde{C}\right] \tag{2.10}$$

where C_1 is an $m \times m$ matrix and $\tilde{C}$ is an $m \times (n-m)$ matrix. We will assume that C_1 and D are non-singular matrices for all q and that upper estimates for the norms of the corresponding inverse matrices (which hold for all q) are known:

$$\left\|C_1^{-1}\right\| \le c_*^{-1}, \quad \left\|D^{-1}\right\| \le d_*^{-1} \tag{2.11}$$

where $c_* > 0$ and $d_* > 0$ are certain constants. Condition (2.11) is satisfied, for example, if the norm of the matrix $\tilde{B}$ is sufficiently small. The corresponding lemma is presented below.

Lemma. Suppose the following inequality is satisfied

$$\tilde{b} < b_* \frac{a_*}{a^*}$$

Then, the matrices C_1 and D from (2.8) are non-singular and the estimates for the norms of the inverse matrices C_1^{-1} and D^{-1}, which occur in inequalities (2.11), are determined by the following equalities

$$c_* = \frac{b_*}{a^*} - \frac{\tilde{b}}{a_*}, \quad d_* = b_* - \tilde{b}\frac{a^*}{a_*}$$

Proof. We shall carry out the proof in parallel for the matrices C_1 and D. The chains of inequalities

$$|C_1 z| \ge |B_1 \alpha_1 z| - |\tilde{B}\tilde{\alpha} z| \ge b_*(a^*)^{-1}|z| - \tilde{b}a_*^{-1}|z| = c_*|z| \ge 0 \tag{2.12}$$

$$|Dz| \ge |B_1 z| - |\tilde{B}A_2^{-1}\tilde{A}z| \ge b_*|z| - \tilde{b}a_*^{-1}a^*|z| = d_*|z| \ge 0 \tag{2.13}$$

hold for any m dimensional vector z.

We will assume that the inverse matrix C_1^{-1} or D^{-1} does not exist. Then, a non-trivial vector λ exists such that $C_1\lambda = 0$ or $D\lambda = 0$. But this is impossible by virtue of the inequalities

$$|C_1\lambda| > 0, \quad |D\lambda| > 0$$

which follow from relations (2.12) and (2.13) when $z = \lambda$ ($\lambda \neq 0$). Hence, the matrices C_1 and D from (2.8) are actually non-singular. We now put $z = C_1^{-1}z$ in relations (2.12) or $z = D^{-1}z'$ in relations (2.13) and obtain the inequalities

$$|C_1^{-1}z'| \le c_*^{-1}|z'|, \quad |D^{-1}z'| \le d_*^{-1}|z'|$$

which hold for any m-dimensional vector z', from which estimates (2.11) follow.

Example of the use of the lemma. We will consider the simplest case when $L = (q_1, \ldots, q_m)$ (see relations (1.12)), that is, when the aim of the control is to bring each of the first m degrees of freedom into the required state with zero velocity. Then, in accordance with notation (2.4), B_1 is an $m \times m$ unit matrix and $\tilde{B}$ is an $m \times (n-m)$ null matrix. The constants b_* and $\tilde{b}$ appearing in estimates (2.6) and (2.7) for the norms of the matrices B_1^{-1} and $\tilde{B}$ can be chosen as $b_* = 1$ and $\tilde{b} = 0$. It is obvious that the condition of the lemma is satisfied in this case. As a result, using the lemma we find c_* and d_* appearing in estimates (2.11) for the norms of the matrices $C_1^{-1}D^{-1}$: $c_* = 1/a^*$ and $d_* = 1$.

We will assume that the second partial derivatives of the function L_i are bounded for all q:

$$\left|\frac{\partial^2 L_i}{\partial q_j \partial q_k}\right| \le c_1$$

where c_1 is a certain constant.

3. Decomposition and the game approach

Differentiating both sides of the second equality of (1.12), we obtain

$$\ddot{x} = B\ddot{q} + R; \quad R = \sum_{j,k=1}^{n} \frac{\partial^2 L}{\partial q_j \partial q_k}\dot{q}_j \dot{q}_k \tag{3.1}$$

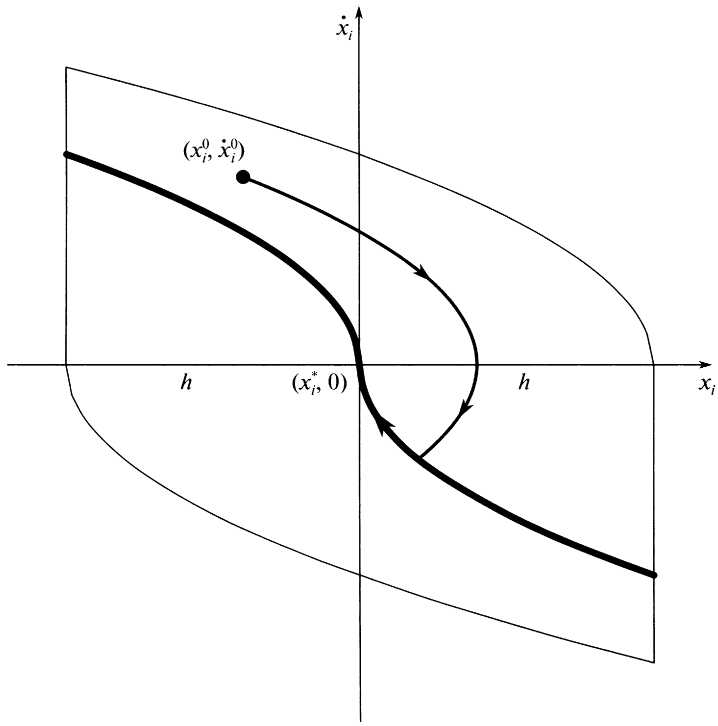

Fig. 1.

Eliminating $\ddot{q}$ from Eqs (1.4) and (3.1), we arrive at the second order differential equations for x

$$\ddot{x} = u + \upsilon; \quad u = CU, \quad \upsilon = CS + R \tag{3.2}$$

We shall consider the m-dimensional vector u as the new control vector and the m-dimensional vector υ as the unknown perturbation. In order to take account of the constraints imposed on the initial control U, we impose the following constraints on the components of the new control u

$$|u_i| \le u_0, \quad u_0 = c_* m^{-1/2} \min_i U_i^0, \quad i = 1,\ldots,m \tag{3.3}$$

We will make a simplifying assumption which we will prove in Sections 4–6. Suppose, during the motion, the moduli of the components of the vector υ are also bounded by a certain constant υ_0:

$$|\upsilon_i| \le \upsilon_0, \quad i = 1,\ldots,n \tag{3.4}$$

and the condition

$$u_0 - \upsilon_0 \ge X \tag{3.5}$$

is satisfied, where $X > 0$ is a certain constant. Then, u_i and υ_i in the i-th equation of (3.2) can be considered as the controls of two players where the possibilities of the control of the first player are greater than the possibilities of the control of the second player. We use the results of the theory of differential games and specify the feedback control $u_i(x_i, \dot{x}_i)$ in the form[2]

$$u_i = \begin{cases} -u_0 \operatorname{sign}(\dot{x}_i - \psi_i^*), & \dot{x}_i \ne \psi_i^* \\ -u_0 \operatorname{sign} \dot{x}_i, & \dot{x}_i = \psi_i^* \end{cases}; \quad \psi_i^*(x_i) = \psi(x_i - x_i^*), \quad \psi(\cdot) = -(2X|\cdot|)^{1/2} \operatorname{sign}(\cdot) \tag{3.6}$$

We note that this control is identical to the time-optimal control for the system

$$\ddot{x}_i = X\frac{u_i}{u_0}, \quad |u_i| u_0, \quad x_i(\tau) = x_i^*, \quad \dot{x}_i(\tau) = 0, \quad \tau \to \min$$

The non-zero components of the required initial control are specified by the relations

$$U_i = [C_1^{-1} u]_i, \quad i = 1,\ldots,m \tag{3.7}$$

in which the components of the vector u are given by relations (3.6) and (1.12). The non-singular matrix C_1 was introduced earlier in (2.10).

We shall assume that parameters X and h exist such that the initial state $x_i^0, \dot{x}_i^0$ of each equation of system (3.2) lies in a corresponding domain Ω_i (see Fig. 1

$$\Omega_i = \left\{(x_i, \dot{x}_i): \quad -h \le x_i - x_i^* h \le x, \quad -\sqrt{2X(x_i - x_i^* + h)} \le \dot{x}_i \le \sqrt{2X(h - x_i + x_i^*)}\right\} \tag{3.8}$$

It is then easy to show that the phase trajectory $x_i(t), \dot{x}_i(t)$ lies as a whole in the domain Ω_i (see Ref. 17).

The time of motion τ_i of the i-th equation of system (3.2) into the state (1.15) depends on the specific realization of the perturbation υ_i and on the initial state $x_i^0, \dot{x}_i^0$. In the case of motion from points of the domain Ω_i which are farthest from the terminal point and in the case of the worst perturbation $\upsilon_i = -(1 - X/u_0)u_i$ which satisfies inequalities (3.4) and (3.5), this time is a maximum:

$$\max \tau_i = 4\sqrt{h/X} \tag{3.9}$$

The velocity which has the largest possible modulus corresponds to these motions

$$\max|\dot{x}_i| = 2\sqrt{Xh} \tag{3.10}$$

We now consider system (3.2) as a whole and denote its largest possible time of motion by τ^*. It is obvious that the time τ^* is equal to the longest possible time of motion (3.9) of each of the equations (3.2):

$$t \le \tau^* = 4\sqrt{h/X} \tag{3.11}$$

Moreover, the inequality

$$|\dot{x}| \le \sqrt{m} \max|\dot{x}_i| = 2\sqrt{mXh} \tag{3.12}$$

follows from relation (3.10).

4. Estimate of the perturbations

We will now verify the initial assumptions (3.4) and (3.5) concerning the boundedness of the perturbations. We will first determine the quantity υ_0 appearing in inequality (3.4). The increment in the velocity vector $\dot{q}$ during the control process

$$\delta\dot{q}(t) = \dot{q}(t) - \dot{q}^0$$

is denoted by $\delta\dot{q}$ and a certain constant (as yet unknown) which sets an upper bound on the modulus of the quantity $\delta\dot{q}$ is denoted by $\Delta\dot{q}$:

$$|\delta\dot{q}(t)| \le |\Delta\dot{q} \tag{4.1}$$

Using the auxiliary relations

$$\sum_{i=1}^{n} |\dot{q}_i| \le \sqrt{n}|\dot{q}| \le \sqrt{n}(|\dot{q}^0| + |\delta\dot{q}|) \le \sqrt{n}(|\dot{q}^0| + \Delta\dot{q}) \tag{4.2}$$

we obtain the chain of inequalities

$$\left|\sum_{j,k=1}^{n} \Gamma_{ijk}\dot{q}_j\dot{q}_k\right| \le \frac{3}{2}c\left(\sum_{j=1}^{n}|\dot{q}_j|\right)^2 \le \frac{3}{2}nc(|\dot{q}^0| + \Delta\dot{q})^2, \quad i = 1,\dots,n$$

$$\left|\sum_{j,k=1}^{n} \frac{\partial^2 L_i}{\partial q_j \partial q_k}\dot{q}_j\dot{q}_k\right| \le c_1\left(\sum_{j=1}^{n}|\dot{q}_j|\right)^2 \le nc_1(|\dot{q}^0| + \Delta\dot{q})^2, \quad i = 1,\dots,n$$

from which the estimates for the moduli of the vectors S and R follow (see expressions (1.5) and (3.1))

$$|S| \le Q^0 + \frac{3}{2}n^{3/2}c(|\dot{q}^0| + \Delta\dot{q})^2, \quad Q^0 = \left(\sum_{i=1}^{n} Q_i^{02}\right)^{1/2}$$

$$|R| \le m^{1/2}nc_1(|\dot{q}^0| + \Delta\dot{q})^2 \tag{4.3}$$

Taking account of relations (4.3) and expression (3.2) for υ, we obtain

$$\upsilon_0 = c^*Q^0 + k(|\dot{q}^0| + \Delta\dot{q})^2; \quad k = n\left(\frac{3}{2}n^{1/2}cc^* + m^{1/2}c_1\right) \tag{4.4}$$

5. The formula for the accelerations of the initial system

We now introduce the vector notation q_1, q_2, S_1, S_2 Here, the vectors q_1 and S_1 are formed from the first m components of the vectors q and S respectively. Similarly, the vectors q_2 and S_2 are formed from the last n - m components of the vectors q and S respectively. When the new notation is taken into account, the last n - m equations of (1.4) take the form

$$\tilde{A}(q)\ddot{q}_1 + A_2(q)\ddot{q}_2 = S_2(q,\dot{q},t)$$

whence we find

$$\ddot{q}_2 = A_2^{-1}(-\tilde{A}\ddot{q}_1 + S_2) \quad (5.1)$$

Substituting the expression found into the equality

$$\ddot{x} = B_1\ddot{q}_1 + \tilde{B}\ddot{q}_2 + R$$

which follows from relation (3.1), we obtain

$$\ddot{x} = D\ddot{q}_1 + \tilde{B}A_2^{-1}S_2 + R \quad (5.2)$$

The non-singular matrix D was introduced earlier in (2.8). Solving Eq. (5.2) for $\ddot{q}_1$ and substituting the result

$$\ddot{q}_1 = D^{-1}(\ddot{x} - R) - D^{-1}\tilde{B}A_2^{-1}S_2 \quad (5.3)$$

into equality (5.1), we obtain

$$\ddot{q}_2 = -A_2^{-1}\tilde{A}D^{-1}(\ddot{x} - R) + \left(A_2^{-1}\tilde{A}D^{-1}\tilde{B}A_2^{-1} + A_2^{-1}\right)S_2 \quad (5.4)$$

We now rewrite expressions (5.3) and (5.4) in the more compact form

$$\ddot{q} = E(\ddot{x} - R) + FS_2; \quad E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \quad F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \quad (5.5)$$

The partitioned matrices E and F consist of the $m \times m$ ($E_1(q)$) and the $(n-m) \times m$ ($E_2(q)$) matrices and the $m \times (n-m)$ ($F_1(q)$) and $(n-m) \times (n-m)$ ($F_2(q)$) matrices respectively. For them, we have the expressions

$$E_1 = D^{-1}, \quad E_2 = -A_2^{-1}\tilde{A}D^{-1}, \quad F_1 = -D^{-1}\tilde{B}A_2^{-1}, \quad F_2 = A_2^{-1}\tilde{A}D^{-1}\tilde{B}A_2^{-1} + A_2^{-1} \quad (5.6)$$

We shall use Eq. (5.5) to obtain an estimate of the modulus of the increment in the velocity vector $\Delta\dot{q}$ during the motion.

6. Estimate of the velocities. Determination of the parameters

We integrate Eq. (5.5) from zero to a certain current instant $t \in [0, \tau]$:

$$\dot{q}(t) - \dot{q}^0 = E(q)\dot{x}(t) - E(q^0)\dot{x}^0 - \int_0^t \sum_{i=1}^n \frac{\partial E(q)}{\partial q_i}\dot{q}_i\dot{x}dt + \int_0^t (FS_2 - ER)dt \quad (6.1)$$

The inequality

$$|\delta\dot{q}| \le 2e\max_t|\dot{x}| + e_1\max_t\sum_{i=1}^n|\dot{q}_i(t)|\max_t|\dot{x}|t + e\max_t|R|t + f\max_t|S_2|t \quad (6.2)$$

in which the constants e, e_1 and f, bounding the norms of the corresponding matrices in expression (6.1)

$$\|E(q)\| \le e, \quad \left\|\frac{\partial E(q)}{\partial q_i}\right\| \le e_1, \quad \|F(q)\| \le f$$

appear, follows from the system of equalities (6.1).

Using the estimates (3.11), (3.12), (4.2), (4.3), (6.2) and

$$|S_2| \le Q_2^0 + \frac{3}{2}m^{1/2}nc(|\dot{q}^0| + \Delta\dot{q})^2, \quad Q_2^0 = \left(\sum_{i=m}^n Q_i^{02}\right)^{1/2}$$

we obtain the inequality

$$|\delta\dot{q}| \le 4e\sqrt{mXh} + 2e_1\sqrt{mnXh}\left(|\dot{q}^0| + \Delta\dot{q}\right)\tau^* + fQ_2^0\tau^* + k_1(|\dot{q}^0| + \Delta\dot{q})^2\tau^*$$

$$k_1 = m^{1/2}n\left(\frac{3}{2}fc + ec_1\right) \quad (6.3)$$

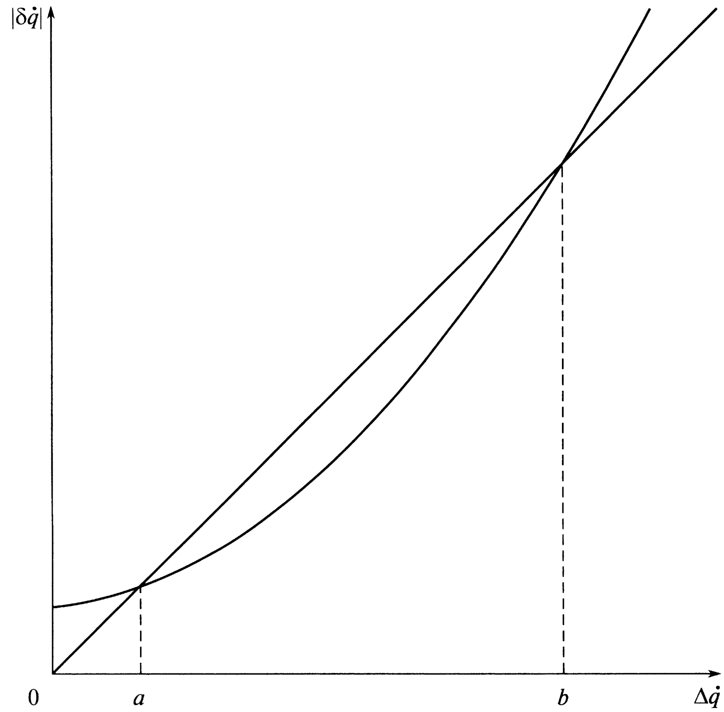

Fig. 2.

We now find the value of $\Delta\dot{q}$ for which inequality (6.3) does not contradict initial inequality (4.1).The linear and quadratic relations corresponding to the equality sign in the above-mentioned relations are shown schematically in Fig. 2. The abscissae of the points of their possible intersection are denoted by the letters a and b. The admissible values of $\Delta\dot{q}$ lie in the interval $[a, b]$.

We equate the right-hand sides of inequalities (4.1) and (6.3), then divide both sides of the resulting equality by $k_1\tau^*$ and substitute expression (3.11) instead of τ^*. We obtain a quadratic equation in $\Delta\dot{q}$

$$(\Delta\dot{q})^2 - 2\left(\kappa(h)\sqrt{X}-|\dot{q}^0|\right)\Delta\dot{q} + \frac{1}{k_1}\left(em^{1/2}X + 2e_1\sqrt{mnXh}|\dot{q}^0| + fQ_2^0\right)+|\dot{q}^0|^2 = 0 \tag{6.4}$$

Equation (6.4) has a positive

$$\kappa(h) = \frac{1}{k_1}\left(\frac{1}{8\sqrt{h}} - e_1\sqrt{mnh}\right) \tag{6.5}$$

Equation (6.4) has a positive solution if its discriminant $Y(X,h)$ is non-negative and the coefficient of $\Delta\dot{q}$ is negative. For this, it is sufficient to require that the following inequalities

$$Y(X,h) = \left(\kappa(h)\sqrt{X}-|\dot{q}^0|\right)^2 - \frac{1}{k_1}\left(em^{1/2}X + 2e_1\sqrt{mnXh}|\dot{q}^0| + fQ_2^0\right)-|\dot{q}^0|^2 \geq 0 \tag{6.6}$$

are satisfied. We will now indicate the method of solving t

$$\kappa(h)\sqrt{X}-|\dot{q}^0| > 0 \tag{6.7}$$

are satisfied. We will now indicate the method of solving the system of inequalities. It follows from inequality (6.7) that the function $\kappa(h)$ must be positive. Taking account of expression (6.5), we rewrite relations (6.6) in the simpler form

$$Y(X,h) = \kappa(h)^2X - \frac{1}{k_1}\left(\frac{1}{4}\sqrt{\frac{X}{h}}|\dot{q}^0| + em^{1/2}X + fQ_2^0\right) \geq 0 \tag{6.8}$$

It is necessary that the coefficient of X should be positive. This leads (when account is taken of the positiveness of the function $\kappa(h)$) to the constraint

$$\kappa(h) > \sqrt{em^{1/2}/k_1}$$

We substitute expression (6.5) instead of $\kappa(h)$ and solve the resulting inequality for h. We have

$$h < \frac{1}{m^{1/2}n}\left(\frac{\sqrt{k_1e + e_1n^{1/2}/2} - \sqrt{k_1e}}{2e_1}\right)^2 \tag{6.9}$$

We now solve the system of inequalities (6.7) and (6.8) for X and obtain

$$X \geq \left[\frac{|\dot{q}^0|}{8h^{1/2}K(h)} + \sqrt{\left(\frac{|\dot{q}^0|}{8h^{1/2}K(h)}\right)^2 + \frac{fQ_2^0}{K(h)}}\right]^2, \quad X > \left(\frac{|\dot{q}^0|}{(h)}\right)^2 \tag{6.10}$$

$$K(h) = k_1\kappa(h)^2 - em^{1/2} \quad (K(h) > 0)$$

We choose the parameters X and h according to inequalities (6.9) and (6.10). We note that inequality (6.9) is satisfied for sufficiently small values of h and inequality (6.10) for sufficiently large values of X. After choosing X and h, the smaller of the two solutions of quadratic equation (6.4) (see Fig. 2) can be taken as the required value of $\Delta\dot{q}$:

$$\Delta\dot{q} = \chi(h)\sqrt{X} - |\dot{q}^0| - \sqrt{Y(X,h)} \tag{6.11}$$

We now substitute expression (4.4) for υ_0 into inequality (3.5) and then replace $\Delta\dot{q}$ in the resulting inequality by expression (6.11). We obtain

$$u_0 \geq X + c^*Q^0 + k\left(\chi(h)\sqrt{X} - \sqrt{Y(X,h)}\right)^2 \tag{6.12}$$

Hence, the initial assumption concerning the boundedness of the perturbations is satisfied in the case of sufficiently large values of u_0, that is, in the case of sufficiently high possibilities of the control.

We will now sum up the results obtained in the form of a theorem.

Theorem. Suppose positive parameters X and h are found which satisfy inequalities (6.9), (6.10) and (6.12). Then, the feedback control $U(q, \dot{q})$, which solves Problem 1, is given by relations (1.12), (3.6) and (3.7). This control brings system (1.4) from initial state (1.13) into set (1.16), if the corresponding initial point $(x^0, \dot{x}^0)$, defined by equalities (1.14), lies in the domain $\Omega = \Omega_1 \times \ldots \times \Omega_m$, where the set Ω_i is given by constraints (3.8). At the same time, the trajectory $(x(t), \dot{x}(t))$ lies in the above-mentioned domain Ω and the time of the control process τ does not exceed the magnitude of τ^*, defined by expression (3.11).

We now present the limiting relation into which the solution of the system of inequalities (6.10) passes when $h \to 0$ (we will only retain the leading terms in the expansion in powers of h)

$$X \geq 64\left[k_1|\dot{q}^0| + \sqrt{\left(k_1|\dot{q}^0|\right)^2 + k_1 fQ_2^0}\right]^2 h \tag{6.13}$$

The expression obtained can be used instead of inequalities (6.10) for choosing the parameter X in the case of very small values of the parameter h: it is assumed that the parameters h is so small that inequality (6.9) is also satisfied.

We now obtain the limiting relations for the right-hand sides of relations (6.11) and (6.12) for small h. Since they depend both on X as well as h, we consider two different situations.

Suppose the parameter X is fixed and h tends to zero:

$$X = \text{const}, \quad h \to 0 \quad (X \gg h)$$

We note that constraint (6.13) (or (6.10)) is automatically satisfied in this case and relations (6.11) and (6.2) take the following form (we only retain the leading terms of the expansion in powers of h)

$$\Delta\dot{q} = 4(em^{1/2}X + fQ_2^0 + k_1|\dot{q}^0|^2)(h/X)^{1/2} \tag{6.14}$$

It can be seen that the estimate for the increment in the v

$$u_0 \geq X + c^*Q^0 + k|\dot{q}^0|^2 \tag{6.15}$$

It can be seen that the estimate for the increment in the velocity (6.14) (or (6.11)) tends to zero. The estimate for the time of the motion (3.11) also tends to zero.

We will now consider the case when the parameter X is directly proportional to h and the two parameters tend to zero:

$$X \sim h, \quad h \to 0$$

To be specific, we will choose the smallest possible value of X corresponding to the equality sign in relation (6.13). In this case, relations (6.11) and (6.12) take the following form (only the leading terms of the expansion in powers of the small parameter are retained)

$$\Delta\dot{q} = \frac{fQ_2^0 + k_1|\dot{q}^0|^2}{2\left[k_1|\dot{q}^0| + \sqrt{\left(k_1|\dot{q}^0|\right)^2 + k_1 fQ_2^0}\right]}$$

$$u_0 \geq c^*Q^0 + k(|\dot{q}^0| + \Delta\dot{q})^2$$

It can be seen that the resulting relations are independent of the parameters X and h. The estimate for the velocity increment is finite and tends to zero. The same can also be said about the estimate for the time of the motion (3.11).

7. Example

We will now consider a double pendulum (Fig. 3) consisting of a fixed base B_0 and two absolutely rigid links B_1 and B_2. The elements of the construction are joined to one another by two ideal cylindrical hinges O_1 and O_2 in such a way that the two links can only execute motions in the vertical plane. The centre of mass C_1 of the link B_1 lies on the ray O_1O_2. The position of the centre of mass C_2 of the link B_2 does not coincide with the position of the hinge O_2. The system is controlled by the moment of the forces M created in the hinge O_1. The friction in the hinges and of air is not taken into account.

The Lagrange equations describing the motion of the system have the form

$$(m_2 l_1^2 + I_1)\ddot{q}_1 + m_2 l_1 l_{g2}\cos(q_2 - q_1)\ddot{q}_2 - m_2 l_1 l_{g2}\sin(q_2 - q_1)\dot{q}_2^2 = M + G_1^0 \sin q_1$$
$$m_2 l_1 l_{g2}\cos(q_2 - q_1)\ddot{q}_1 + I_2\ddot{q}_2 + m_2 l_1 l_{g2}\sin(q_2 - q_1)\dot{q}_1^2 = G_2^0 \sin q_2$$
$$G_1^0 = g(m_1 l_{g1} + m_2 l_1), \quad G_2^0 = g m_2 l_{g2} \tag{7.1}$$

The following notation has been introduced here: q_i is the angle between the straight line O_iC_i and the vertical axis, l_{gi} is the length of the segment O_iC_i, l_1 is the length of the segment O_1O_2, m_i is the mass of the link B_i, I_i is the moment of inertia of the link B_i with respect to the axis of the hinge O_i, $G_i^0 \sin q_i$ is the torque created by the force of gravity in the i-th hinge and g is the acceleration due to gravity.

The constraint

$$|M| \le M_0 \tag{7.2}$$

where M_0 is a given constant, is imposed on the control torque.

We now change to the new variable t' and introduce the dimensionless parameters α and β

$$t' = \left(\frac{1}{m_2 l_1 l_{g2}}\right)^{1/2} t, \quad \alpha = \frac{I_1 + m_2 l_1^2}{m_2 l_1 l_{g2}}, \quad \beta = \frac{I_2}{m_2 l_1 l_{g2}} \tag{7.3}$$

The prime on the new variable t' will henceforth be omitted. Equations (7.1) take the form

$$\alpha\ddot{q}_1 + \cos(q_2 - q_1)\ddot{q}_2 = M + G_1^0 \sin q_1 + \sin(q_2 - q_1)\dot{q}_2^2$$
$$\cos(q_2 - q_1)\ddot{q}_1 + \beta\ddot{q}_2 = G_2^0 \sin q_2 - \sin(q_2 - q_1)\dot{q}_1^2 \tag{7.4}$$

We demonstrate the calculation of the control parameters for the following characteristics of system (7.1)

$$l_1 = l_2 = 1\ \text{m}, \quad l_{g1} = l_{g2} = 0.5\ \text{m}, \quad I_1 = I_2 = 0.3333\ \text{kg}\cdot\text{m}^2$$
$$m_1 = m_2 = 1\ \text{кг}, \quad M_0 = 120\ \text{N}\cdot\text{m}, \quad g = 9.81\ \text{ms}^{-2}$$

The following parameters of system (7.4) correspond to the selected values

$$\alpha = 2.667, \quad \beta = 0.6667, \quad G_1^0 = 14.72, \quad G_2^0 = 4.905$$

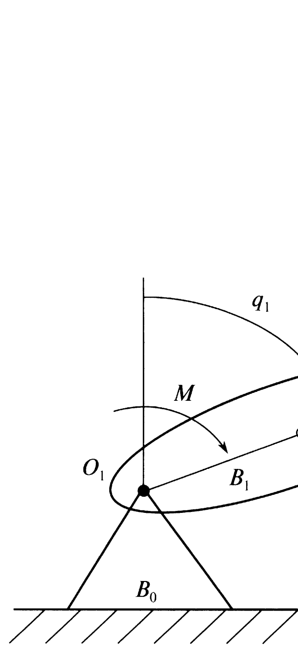

Fig. 3.

We specify, for example, the function L (1.12) in the form

$$x = L(q) = q_2 - q_1$$

In this case, the angle between the links emerges as the x coordinate. We will consider the problem of controlling the configuration of the pendulum when it is required to change the angle x from a certain initial value H to a final value $-H$, where $H < \pi/2$ is a certain sufficiently small positive constant. The initial position of the pendulum is arbitrary. However, for simplicity, we shall assume that the initial angular velocity of rotation of the two links is equal to zero. This auxiliary problem arose[7] when solving the more complex problem of the swinging of a double pendulum.

At the initial instant, the pendulum is at rest and, consequently, $|\dot{q}^0| = 0$. Moreover, by virtue of the linearity of the function $L(q)$, the modulus of the vector R as well as the constant c_1 are equal to zero (see relations (3.1) and (4.3)). In this case,

$$k = \frac{3}{2}n^{3/2}cc^*, \quad k_1 = \frac{3}{2}m^{1/2}nfc \quad (m = 1, n = 2)$$

and the first inequality of (6.10) is simplified:

$$X \geq \frac{fQ_2^0}{k_1\kappa(h)^2 - em^{1/2}} \tag{7.5}$$

and the second equality of (6.10) is automatically satisfied, and relations (6.11) and (6.12) take the form

$$\Delta\dot{q} = \chi(h)\sqrt{X} - \sqrt{\chi(h)^2 X - \frac{1}{k_1}(em^{1/2}X + fQ_2^0)} \tag{7.6}$$

$$u_0 \geq X + c^*Q^0 + k(\Delta\dot{q})^2 \tag{7.7}$$

The kinetic energy matrix has the form

$$A = \begin{bmatrix} \alpha & \cos(q_2 - q_1) \\ \cos(q_2 - q_1) & \beta \end{bmatrix} \tag{7.8}$$

The matrix B from (1.12) is constant:

$$B = [-1 \quad 1] \tag{7.9}$$

Using expressions (7.8) and (7.9), we find the matrix C (2.8):

$$C = \begin{bmatrix} -\frac{\beta + \cos(q_2 - q_1)}{\alpha\beta - \cos^2(q_2 - q_1)} & \frac{\alpha + \cos(q_2 - q_1)}{\alpha\beta - \cos^2(q_2 - q_1)} \end{bmatrix}$$

The matrices E_1, E_2, F_1, F_2 (5.6) and D (2.8) are scalar functions:

$$E_1 = -\frac{\beta}{\beta + \cos(q_2 - q_1)}, \quad E_2 = \frac{\cos(q_2 - q_1)}{\beta + \cos(q_2 - q_1)}$$

$$F_1 = F_2 = \frac{1}{\beta + \cos(q_2 - q_1)}, \quad D = -\frac{\beta + \cos(q_2 - q_1)}{\beta}$$

where

$$\frac{\partial E_1(q)}{\partial q_1} = \frac{\partial E_2(q)}{\partial q_1} = -\frac{\partial E_1(q)}{\partial q_2} = -\frac{\partial E_2(q)}{\partial q_2} = \frac{\beta\sin(q_2 - q_1)}{[\beta + \cos(q_2 - q_1)]^2}$$

It follows from the monotonicity of the change in the coordinate $x(t)$ (when $\dot{x}(0) = 0$) that the angle between the links will be bounded $|q_2 - q_1| \leq H < \pi/2$ in the control process, that is,

$$\cos(q_2 - q_1) \geq \cos H, \quad |\sin(q_2 - q_1)| \leq \sin H$$

To be specific, we let $H = 0.025$, and, in this case, it can be assumed that $h = 2H = 0.05$.

We now determine the values of the constants Q_2^0, c, e, e_1, f, required to solve inequalities (6.9) and (7.5):

$$Q_2^0 = G_2^0 = 4.905, \quad c = \sin H = 0.02500$$

$$\|E(q)\| \leq \frac{\sqrt{\beta^2 + \cos^2(q_2 - q_1)}}{\beta + \cos(q_2 - q_1)} \leq e = \frac{\sqrt{\beta^2 + 1}}{\beta + 1} = 0.7211$$

$$\left\|\frac{\partial E(q)}{\partial q_i}\right\| \le e_1 = \frac{\sqrt{2}\beta \sin H}{[\beta + \cos H]^2} = 0.008488$$

$$\|F(q)\| \le f = \frac{\sqrt{2}}{\beta + \cos H} = 0.8487$$

In the case of the values of the constants which have been found, inequalities (6.9) and (7.5) take the form

$$h < 0.3199, \quad X \ge \frac{16.96}{\left(h^{-1/2} - 0.09603h^{1/2}\right)^2 - 2.937} \tag{7.10}$$

It can be seen that the value of the parameter $h = 0.05$, chosen earlier, satisfies the first inequality of (7.10). Substituting this value into the right-hand side of the second inequality of (7.10), we find the admissible values of he parameter X corresponding to the chosen value of h:

$$X \ge 1.005$$

We put

$$X = 110$$

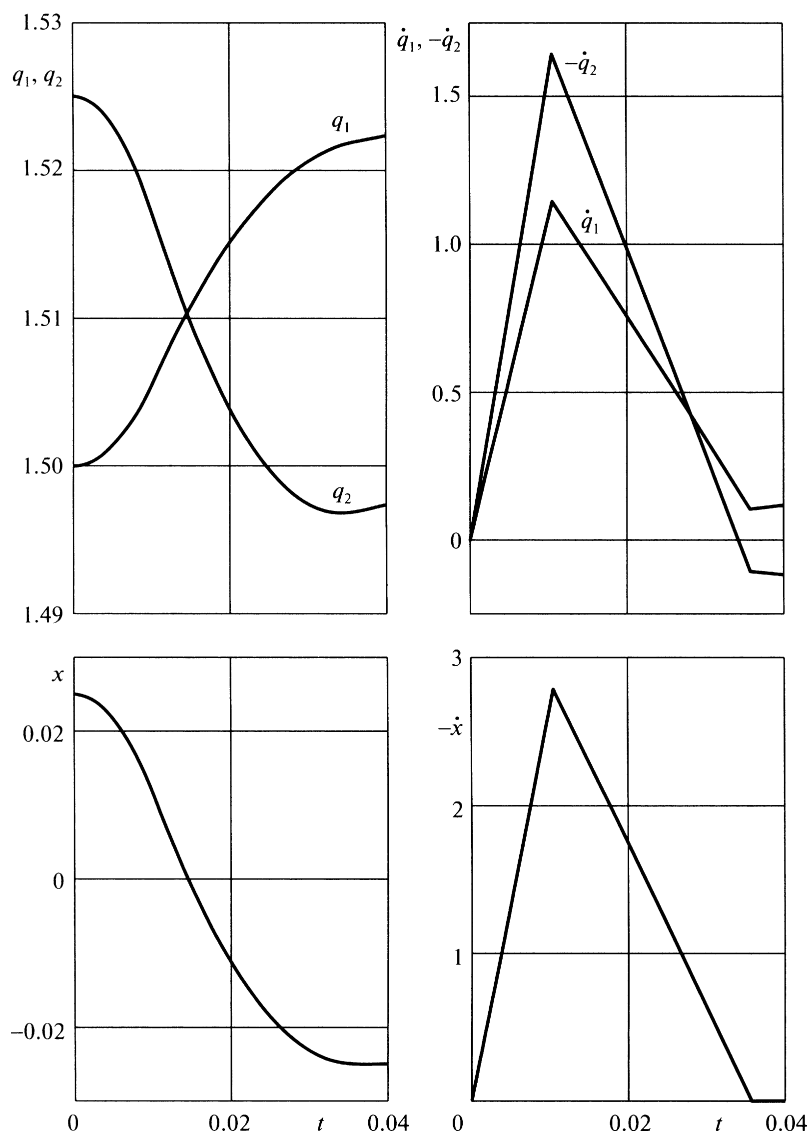

Fig. 4.

We now determine the constants Q^0 and c^*, used in inequality (7.7), and also the constant c_*, used in equality (3.3):

$$Q^0 = \sqrt{G_1^{02} + G_2^{02}} = 15.51$$

$$c^* = \frac{\sqrt{\alpha^2 + 2(1+\alpha+\beta) + \beta^2}}{(\alpha\beta - 1)^2} = 6.658, \quad c_* = \frac{\beta + \cos H}{\alpha\beta - \cos^2 H} = 2.141$$

For the values of all the constants which have been found and the chosen parameters X and h, inequality (7.7) reduces to the constraint

$$u_0 \geq 252.5 \tag{7.11}$$

and, as a result (when equality (3.3) and inequality (7.7) are taken into account), to the constraint on the admissible amplitude of the control torque:

$$M_0 = u_0/c_* \geq 117.0$$

The initial parameters of the pendulum satisfy this constraint.

8. Numerical modelling

We will now present the results of numerical modelling The initial conditions were chosen to be as follows:

$$q_1^0 = 1.5 \text{ рад}, \quad q_2^0 = 1.525 \text{ рад}, \quad \dot{q}_1^0 = \dot{q}_2^0 = 0$$

It is required to bring the system into the terminal set

$$q_2 - q_1 = x^* = -0.025 \text{ рад}, \quad \dot{q}_2 - \dot{q}_1 = 0$$

Equations (7.4) were integrated in the case of the control

$$M = M_0 c_* \frac{\alpha\beta - \cos^2 x}{\beta + \cos x} \begin{cases} \operatorname{sign}(\dot{x} - \psi^*), & \dot{x} \neq \psi^* \\ \operatorname{sign}\dot{x}, & \dot{x} = \psi^* \end{cases}$$

$$\psi^*(x) = \psi(x - x^*), \quad \psi(\cdot) = -(2X|\cdot|)^{1/2}\operatorname{sign}(\cdot); \quad x = q_2 - q_1, \quad \dot{x} = \dot{q}_2 - \dot{q}_1$$

constructed using the proposed technique (the values of M, c_*, α, β and X were found in the preceding section).

The quantities $q_1, \dot{q}_1, q_2, \dot{q}_2, x, \dot{x}$ are shown in Fig. 4 as functions of the time t, and the relation $\dot{x}(x)$, where $x = q_2 - q_1$ is the angle between the links, is shown in Fig. 5. The three basic stages of the control process, corresponding to bringing of the point $(x, \dot{x})$ onto the switching

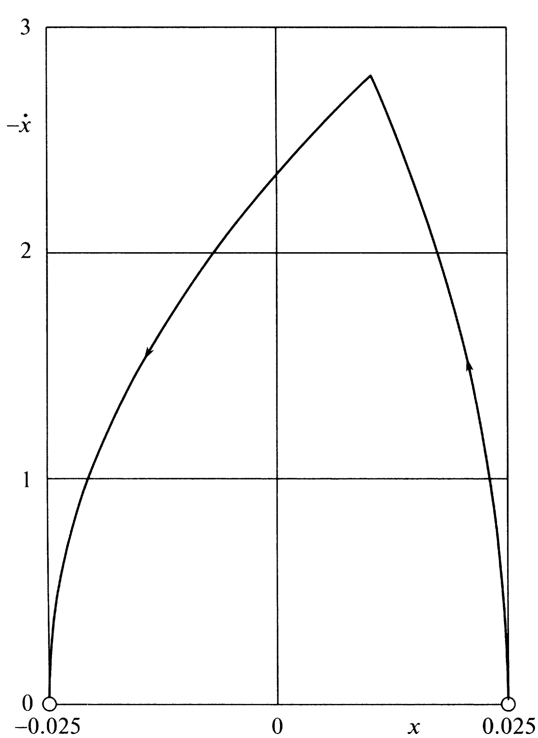

Fig. 5.

curve $\psi^*(x)$, its motion along the switching curve and retention at the terminal point $(x^*,0)$, are clearly expressed in Fig. 4. Only the first two stages are seen in Fig. 5. The control time was found to be equal to 0.03586 while, in the case of its estimate (3.11), it was found to be equal to 0.08528.

The maxima of the quantities $|\dot{x}|$ and $|\dot{q}|$ were attained for $t = 0.01051$ and they are

$$\max|\dot{x}| = 2.789, \quad \max|\dot{q}| = 2.003$$

According to formulae (3.12) and (7.6), the upper limits for the above-mentioned quantities have the form

$$\max|\dot{x}| \leq 4.690, \quad \max|\dot{q}| \leq 7.457$$

Acknowledgements

This research was supported financially by the Russian Foundation for Basic Research (07-01-92109, 08-01-00411, 08-08-00292), the Russian Science, Support Foundation and the Programme for the Support of Leading Scientific Schools (NSh-4315.2008.1).

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Translated by E.L.S.